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A constitutive model for metals applicable at high-strain rate

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A model, applicable at high-strain rate, is presented for the shear modulus and yield strength as functions of equivalent plastic strain, pressure, and internal energy (temperature). The parameters needed to implement the model have been determined for 14 metals. Using this model, hydrodynamic computer simulations have been successful in reproducing measured stress and free-surface-velocity-vs-time data for a number of shock-wave experiments.

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I. INTRODUCTION

In hydrodynamic computer codes, e.g., KO or HEMP, stress-tensor components are split into a hydrostatic equation of state and a modified elastic-perfectly plastic constitutive model. In one modification, the yield strength \( Y \) (in the Von Mises sense) increases with increasing plastic strain. Models of this effect, called work hardening, have been proposed previously. For example, Wilkins and Guinan have described a work-hardening model for OFHC copper and 6061-T6 aluminum.

However, \( Y \) and the shear modulus \( G \) also increase with increasing pressure and decrease with increasing temperature. It is the purpose of this paper to describe our modifications to the elastic-perfectly plastic constitutive model that account for the effects of pressure \( P \), temperature \( T \), and equivalent plastic strain \( \varepsilon \), on \( Y \) and \( G \), and to determine the parameters needed to implement this model for 14 metals.

While \( Y \) also increases with increasing strain rate \( \dot{\varepsilon} \), it does not seem reasonable to expect it to do so without limit. Therefore, we assume that a value of \( \dot{\varepsilon} \) exists beyond which strain rate has a minimal effect on \( Y \). To check this assumption, we examined experimental shock-induced free-surface-velocity-vs-time records. This examination showed that rate-dependent effects appear to play a major role in determining the shape of these shock-wave profiles at stresses of a few gigapascals. However, at stresses approaching 10 GPa, these effects become insignificant and the data can be successfully reproduced with a rate-independent model. In terms of \( \dot{\varepsilon} \), these data indicate that our assumption of a rate-independent constitutive model is valid for \( \dot{\varepsilon} \lesssim 10^5 \text{ s}^{-1} \).

The rapid decrease of rate-dependent effects with increasing dynamic stress may be explained by the increase in temperature with increasing stress. In liquids, rate-dependent effects, such as viscosity, appear to decrease exponentially with temperature; a similar strong temperature dependence may exist for rate-dependent effects in shocked solids.

While there are extensive data showing the pressure and temperature variation of \( G \), there are no such definitive data for \( Y \). However, for materials that do not exhibit strain-rate dependence, \( Y \) is expected to be proportional to \( G \); i.e., \( Y/G \) is a constant. Generally, \( Y/G \) decreases with temperature at a rate directly related to \( \dot{\varepsilon} \), but with less temperature dependence at higher \( \dot{\varepsilon} \). However, the assumption that \( Y/G \) is a constant is consistent with the neglect of strain-rate effects. The intimate connection between the \( T \) and \( \dot{\varepsilon} \) dependences of \( Y/G \) requires that both be included in any self-consistent treatment; consequently, the temperature dependence of \( Y \) is assumed to be the same as that of \( G \). These points are further discussed by McClintock and Argon. In addition, the recent data of Gather and Walton on the radial collapse of electrically heated hollow tantalum rods in an isobaric environment show \( Y(T) \) to be consistent with this assumption.

Experimental evidence indicates that \( Y \) may increase more rapidly with pressure than does \( G \). However, the accuracy of these data is not sufficient to warrant using a form for the pressure dependence of \( Y \) that is different from that used to represent the pressure dependence of \( G \). Indeed, recent theoretical and experimental evidence indicates that

\[
\frac{1}{Y} \frac{dY}{dP} \approx \frac{1}{G} \frac{dG}{dP}.
\]

In Sec. II, we describe our constitutive model and show that it is reasonably consistent with the known data. In Sec. III, the parameters needed to implement the model are listed for 14 metals. In Sec. IV, hydrodynamic computer simulations using this model are discussed. These were successful in reproducing measured stress and free-surface-velocity-vs-time records for a number of shock-wave experiments.

II. THE CONSTITUTIVE EQUATIONS

The constitutive relations for \( G \) and \( Y \) as functions of \( \varepsilon \), \( P \), and \( T \) for high \( \dot{\varepsilon} \) are

\[
G = G_0 \left[ \left( \frac{G^*}{G_0} \right)^{1/3} + \frac{G^*}{G_0} (T-300) \right],
\]

\[
Y = Y_0 \left[ \frac{1 + \beta (\varepsilon + \epsilon_i)}{Y_0} \right]^{\eta/3} \left[ \frac{1 + \left( \frac{Y^*}{Y_0} \right)^{1/3} + \left( \frac{G^*}{G_0} \right) (T-300) \right],
\]

subject to the limitation that

\[
Y = Y_0 \left[ \frac{1 + \beta (\varepsilon + \epsilon_i)}{Y_0} \right]^{\eta/3} \leq Y_{\text{max}}.
\]

Here, \( \eta \) is compression, defined as the initial specific volume \( \nu_0 \) divided by the specific volume \( \nu \), \( \beta \) and \( n \) are work-hardening parameters, and \( \varepsilon_i \) is the initial equivalent plastic strain, normally equal to zero. The subscript \( 0 \) refers to the reference state \( (T = 300 \text{ K}, P = 0, \varepsilon = 0) \). Primed parameters
with the subscripts $P$ and $T$ imply derivatives of that parameter with respect to pressure or temperature at the reference state.

Because many hydrodynamic computer codes such as KO or HEMP operate with energy $E$ rather than temperature, $T$ is defined as the difference between the total energy and the energy along the zero Kelvin isotherm $E_c(1)$ divided by the specific heat of the solid $C$, i.e., $T = (E - E_c)/C$. The quantity $E_c(1)$ is equal to the energy along the zero Kelvin isotherm and is defined as the difference between the total energy and the energy corresponding to the temperature along that adiabat.

The quantity $E_c(1)$ is limited to $G_{\gamma}/G_{\gamma_0}$, which is the gas constant $C$ and the specific heat of the solid $C$, i.e., $E_c(1) = \frac{1}{2} \varepsilon T$.

At low pressures ($P \leq 2$ GPa), experiments show that $G$ varies linearly with $P$. At ultrahigh pressures, Thomas-Fermi theory states the $P$ is proportional to $\eta^{2/3}$ and $G$ to $\eta^{4/3}$. In Eq. (1) it is apparent that $G = G_0 + G_{\gamma}P$ as $P \to 0$ and $\eta \to 1$, and that $G$ will vary as $\eta^{4/3}$ as $\eta \to \infty$.

In addition, the pressure-dependent part of Eq. (1), with measured $G_0$ and $G_{\gamma}$, has been compared with Thomas-Fermi calculations of $G$ at ultrahigh pressures. Equation (1) predicts $G$ within a factor of 2–3 of the Thomas-Fermi values. Of equal importance is the fact that they are lower, with the exceptions of $B$ and $U$. Therefore, in Eq. (1), $G$ will not become unreasonably large at large $P$. Consequently, because it has the appropriate limiting behavior at both high and low pressure, Eq. (1) appears a reasonable and simple way of representing pressure dependence.

However, to allow for the possibility that $Y$ may increase more rapidly with $P$ than does $G$, the pressure coefficients for each can be set individually. If $Y_{\gamma}/Y_{\gamma_0}$ is greater than $G_{\gamma}/G_{\gamma_0}$, $Y$ is limited to $10 G$ so that it cannot become unphysically large.

Experiments show that for many important engineering materials, $G$ decreases nearly linearly with $T$. However, $G$ has been measured to approximately the melt temperature $T_m$, for only a limited number of materials. Of particular interest are $Bi$ and $Sn$, $In$ and $Pb$, because data exist to within 0.1 K of $T_m$. Also, there are good data for $Al$, $Cd$, and $Zn$ to within 8, 19, and 23 K of $T_m$, respectively.

Table 1. Numerical values of parameters used in the constitutive equations.

<table>
<thead>
<tr>
<th>Material</th>
<th>$G_0$ (GPa)</th>
<th>$Y_0$ (GPa)</th>
<th>$Y_{\max}$ (GPa)</th>
<th>$\beta$</th>
<th>$n$</th>
<th>$G_{\gamma}/G_0$ (TPa$^{-1}$)</th>
<th>$-G_{\gamma}/G_0$ (kK$^{-1}$)</th>
<th>$T_{m}$ (K)</th>
<th>$\gamma_0$</th>
<th>$a$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al (6001-T6)</td>
<td>27.6</td>
<td>0.29</td>
<td>0.68</td>
<td>125</td>
<td>0.10</td>
<td>1.0</td>
<td>0.62</td>
<td>1.220</td>
<td>1.97</td>
<td>1.5</td>
<td>$T_{m}$ and $a$ are for the 2024 alloy</td>
</tr>
<tr>
<td>Au</td>
<td>28.0</td>
<td>0.02</td>
<td>0.225</td>
<td>49</td>
<td>0.39</td>
<td>0.38</td>
<td>0.31</td>
<td>1.970</td>
<td>2.99</td>
<td>2.4</td>
<td></td>
</tr>
<tr>
<td>Be (S-200)</td>
<td>15.1</td>
<td>0.33</td>
<td>1.23</td>
<td>81</td>
<td>0.22</td>
<td>0.22</td>
<td>0.26</td>
<td>1.820</td>
<td>1.11</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>Cu (OFHC half hard)</td>
<td>47.7</td>
<td>0.12</td>
<td>0.64</td>
<td>36</td>
<td>0.45</td>
<td>0.28</td>
<td>0.38</td>
<td>1.790</td>
<td>2.02</td>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>Mg</td>
<td>16.5</td>
<td>0.19</td>
<td>0.48</td>
<td>1100</td>
<td>0.12</td>
<td>0.12</td>
<td>0.51</td>
<td>1.150</td>
<td>1.54</td>
<td>1.2</td>
<td>$G_{\gamma}$ and $G_{\gamma}$ are for pure Mg</td>
</tr>
<tr>
<td>(AZ31B-H24)</td>
<td>37.7</td>
<td>0.70</td>
<td>1.4</td>
<td>5</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nb (Annealed)</td>
<td>85.5</td>
<td>0.14</td>
<td>1.2</td>
<td>46</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Pb</td>
<td>8.6</td>
<td>0.008</td>
<td>0.1</td>
<td>110</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Pt (Annealed)</td>
<td>63.7</td>
<td>0.03</td>
<td>0.34</td>
<td>1300</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Stainless steel (304)</td>
<td>77.0</td>
<td>0.34</td>
<td>2.5</td>
<td>43</td>
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<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Ta</td>
<td>69.0</td>
<td>0.77</td>
<td>1.10</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ti</td>
<td>43.4</td>
<td>0.71</td>
<td>1.45</td>
<td>780</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>U</td>
<td>86.7</td>
<td>0.40</td>
<td>1.68</td>
<td>2000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>W</td>
<td>160.0</td>
<td>2.2</td>
<td>4.0</td>
<td>(7.7)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

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these cases, either $G$ is nearly linear with $T$ or at least the rate of change of $G$ with $T$ is not substantially greater at $T_m$ than at lower $T$. For some materials, such as alpha uranium, a $T^2$ term is definitely needed to accurately reproduce the data. However, because the temperature-dependent term in $G$ is typically $\sim 10\%$ of the pressure term, we assume that an average linear dependence up to $T \approx T_m$ will suffice in Eq. (1) for all materials. For the reasons given in Sec. I, this form of temperature dependence has also been applied to the yield strength.

The work-hardening function $\{1 + \beta (\epsilon + \epsilon_s)\}$ is a semiempirical one that appears to do the best overall job of fitting the data for our 14 metals. If high $- \epsilon$ data were lacking, we used the highest $- \epsilon$ or the lowest-temperature data that was felt to most nearly approximate dynamic conditions. In some cases, such as stainless steel or Be, this two-parameter form fits the data very well over a wide range of $\epsilon$. In other cases, notably uranium, a compromise fit was necessary to do a reasonable job over a modest variation in $\epsilon$. This work-hardening model is consistent with the work of Wilkins and Guinan and Honodel, who determined the yield strength of many materials through computer simulation of cylinder-deceleration experiments. Average $\epsilon$ for their experiments was $\sim 10^6$ s$^{-1}$.

Some materials, notably U and Cu, are extremely sensitive to their previous mechanical history. The yield strength of annealed U is zero, while in shock-wave experiments $Y_0$ has varied from 0.4 to 1.1 GPa. We have assumed that any material will follow a unique work-hardening curve; therefore, prior treatment, e.g., rolling or machining, will result in the material's being at a higher initial stress level at an effective equivalent plastic strain of $\epsilon$. Our uranium parts, as used in a variety of experiments, were in a highly worked state. Therefore, setting $\epsilon_s = 0.0375$ gives $Y_0 = 0.8$ GPa, which corresponds to the $Y_0$ we most typically observed.

It is necessary to have some means of turning off material strength at melt, but the discussion of melt models is outside the scope of this paper. However, for the sake of completeness, in Appendix A we will describe a melt model that is in several of our hydrodynamic computer codes.

III. NUMERICAL VALUES FOR THE PARAMETERS IN THE CONSTITUTIVE RELATIONS

Table I lists the values of the parameters used in Eqs. (1)-(4). Except where noted otherwise in square brackets, all values of $G_p$ and $G_e$ are from Guinan and Steinberg from the data of Royce and Isbell; and $G_0$ from Simmons and Wang. Values of $a$ were determined by setting our $Y_0 = a$ equal to that of Royce. Values shown in parentheses are from Royce—see Appendix A—and $G_0$ from Isbell, et al. The ideal sample material is therefore one with a large yield strength at the Hugoniot elastic limit and also one which exhibits significant work hardening. In addition, the material should have large $P$ and $T$ coefficients. This means $G_p$ and $G_e$, but also a small $G_0$. However, relative to the bulk modulus, $G_0$ should be large so that there will be ample separation in time between the elastic and plastic waves. It is difficult to find a material that meets these criteria. For example, copper work hardens significantly, but its initial yield strength is very small. On the other hand, tantalum has a large $Y_0$, but work hardens only slightly.

The ideal sample material is therefore one with a large yield strength at the Hugoniot elastic limit and also one which exhibits significant work hardening. In addition, the material should have large $P$ and $T$ coefficients. This means large $G_p$ and $G_e$, but also a small $G_0$. However, relative to the bulk modulus, $G_0$ should be large so that there will be ample separation in time between the elastic and plastic waves. It is difficult to find a material that meets these criteria. For example, copper work hardens significantly, but its initial yield strength is very small. On the other hand, tantalum has a large $Y_0$, but work hardens only slightly.

The metal we know that best meets these requirements is alpha-phase plutonium. Alpha-plutonium has the largest known value of $G_p/G_0$ and $G_e$. As well as a large $G_e$.
FIG. 1. Experimental (---) and calculated (—) shock-induced wave profiles showing the motion of the aluminum-PMMA interface versus time. The series illustrates the effect of adding to the calculation, step-by-step, the various \( P \), \( T \), and \( \epsilon \)-dependent terms in the constitutive model. (a) Pure hydro, (b) adding a constant \( Y \) and \( G \), (c) adding work hardening, (d) adding the \( P \) dependence of \( G \), (e) adding the \( P \) dependence of \( Y \), (f) adding \( T \) dependence, (g) adding the Bauschinger model with \( G_1 = G_o \), (h) the Bauschinger model with \( G_1 = 0.725G_o \).

and significant work hardening. \( G_o \) is also equal to 82% of the bulk modulus, an atypically large percentage. Our material sciences group has performed 10 experiments on alpha-plutonium at stresses between \(-3\) an 12 GPa. To avoid any bias that might exist with a single kind of detector, both Manganin and capacitor-gauge instrumented experiments were performed.

In calculating these experiments, each principal feature of the wave profiles was predominately affected by specific terms in Eqs. (1) and (2). The difference in the arrival times of the elastic and plastic waves \( \Delta t \) is an extremely sensitive test of any model hydrodynamic calculation. \( \Delta t \) is strongly affected by work hardening and by the pressure dependence of \( Y \). The same is true for the maximum stress. On the other hand, the arrival time of the release wave is affected principally by the pressure dependence of \( G \).

Using independently determined parameters in the constitutive model and a Grüneisen hydrodynamic equation of state, \( \Delta t \) was calculated for several experiments at \(-6\) and 12 GPa. The results were essentially indistinguishable from the measured data. If work hardening and the pressure dependence of the yield strength were omitted from the model, the calculated \( \Delta t \) was approximately twice the experimental value. At 12 GPa, when the temperature dependence was omitted, the calculated and experimental \( \Delta t \) differed by amounts greater than the experimental error.

Finally, the parameter \( G_2/G_o \) is so large for alpha-plutonium that the effect of pressure on \( G \) can be clearly seen even at stresses as low as 3 GPa. Only by including this pressure dependence could we calculate the arrival time of the initial release within experimental error.

After \( \alpha \)-Pu, the metals most likely to show the effect of
$P$, $T$, and $\epsilon$ on $Y$ and $G$ are aluminum and magnesium. We are fortunate that shock-release data on Al have recently become available. In this experiment, a 2.441-mm-thick 6061-T6 Al flyer backed by an ~25-mm-thick polycarbonate sabot impacts a 6.007-mm-thick 6061-T6 Al target at a velocity of 3.82 mm/$\mu$s. A PMMA window, ~25 mm thick, is placed against the free surface of the Al. A velocity interferometer monitors the motion of the Al-PMMA interface as a function of time. Peak stress is about 41.2 GPa.

The experimental data are shown in Fig. 1. In addition, Fig. 1 illustrates the effect of adding to the calculation, step-by-step, the various $P = T$, $T$ and $\epsilon$-dependent terms of the constitutive model. We have used a Grüneisen equation of state with a shock-velocity–particle-velocity $(u_s - u_p)$ equation $u_s = 0.524 + 1.40u_p$ cm/$\mu$s. The form of the Grüneisen gamma was described in Sec. II, and the parameters $\gamma_0$ and $a$ are given in Table I.

Figure 1(a) compares the experimental data with a pure-fluid calculation. The elastic release, extending from about 1.0 to 1.2 $\mu$s, is entirely lacking in the calculation.

Figure 1(b) shows the effect of adding a simple constitutive model to the calculation. Here $Y$ and $G$ are constants equal to 0.29 and 27.6 GPa, respectively. Neither the starting time nor the magnitude of the elastic release is correctly calculated. Even the addition of work hardening does not bring the calculation into accord with the data [Fig. 1(c)].

Figure 1(d) shows the effect of adding pressure dependence to the shear modulus. The onset of the calculated elastic release is now in good agreement with experiment. By adding this pressure dependence to the yield strength we can also calculate the magnitude of this release, as shown in Fig. 1(e). Finally, in Fig. 1(f), we see the effect of adding temperature dependence to both $G$ and $Y$. The time scale is sufficiently expanded so that one can see the small improvement in the calculated starting time of the elastic release; the effect on $Y$ is not apparent on the pressure scale used.

At peak stress, the calculation shows that the independent variables, $P$, $T$, and $\epsilon$ are 40.4 GPa, 1,060 K, and 0.173, respectively. This results in a yield strength of 1.16 GPa.

In order to calculate the shape of the elastic release correctly, we need to add the Bauschinger effect to our constitutive model. The Bauschinger effect is that elastic-plastic materials behave differently upon stress unloading and reverse loading than when they are stress loaded. The Bauschinger model we use has been described in another paper from our material studies group. Consequently, a complete discussion of it is outside the scope of this paper. A brief description is given Appendix B.

In Fig. 1(g), a version of the Bauschinger model, which requires only $G_0$, as an input parameter, has been added to the calculation. The agreement with the experiment is probably well within the accuracy of the data. However, an adjustment was made to $G$, (see Appendix B for definition), setting it equal to ~three-fourths of $G_0$, or 0.200. This results in the small improvement shown in Fig. 1(h).

Figure 1(h) also shows that we calculate the final stress-release state set by the polycarbonate sabot, yet we miss the timing there by ~50 ns or 3.6%. This is well outside the estimated experimental error which is ±10 ns. Considering that we are able to calculate the elastic release so exactly and are able to reproduce complete release profiles for other experiments, it is surprising that the plastic release is not reproduced so well.

In an attempt to improve the agreement between our calculation and the data, we made major changes in the equations of state of the sabot and the window materials—even removing the sabot from the problem—all with negligible effect. We also made reasonable variations in the aluminum equation of state, such as the $u_s - u_p$ equation, the Grüneisen gamma, and the constitutive model itself, yet we could make no significant changes in the calculated plastic release. In addition, a strain-rate-dependent constitutive model, which calculates other experiments at lower stress levels, was tried without success.

The explanation may lie with the experimental data. For example, the passage through the PMMA window of the elastic release may influence the measurement of subsequent waves by changing the optical properties of the window. In addition, the dip in the experimental interface velocity at 1.4 $\mu$s does not seem physically reasonable.

V. SUMMARY

In order to calculate the relative timing between various waves in a shock-wave experiment, the pressure and temperature dependence of the shear modulus must be accounted for in any elastic-plastic constitutive model. Our model does this by including, over a wide range of $P$ and $T$, the considerable experimental and theoretical information regarding $G$.

While there is less information regarding the yield strength at high stress, we have used the available experimental data at low stress, along with reasonable extrapolation models, to improve our ability to predict $Y$ at high $P$ and $T$.

At high stress, the effect of $P$ and $T$ on $Y$ can be even more important than work hardening. For example, aluminum shocked to 41.2 GPa shows that the combined effect of $P$ and $T$ is to increase $Y$ by a factor of ~3, while work hardening only increases $Y$ by ~37%.

ACKNOWLEDGMENTS

The authors thank Dr. Richard Grover for many helpful discussions. We also thank Dr. L. Chhabildas and Dr. J. Asay for making their data available prior to publication. Work was performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore Laboratory under contract number W-7405-ENG-48.

APPENDIX A: MELT MODEL

We use an expression for the melt energy $E_m(\eta)$, taken from the GRAY equation of state, which is based on a modified Lindemann law. Though there are no adequate data on shock-induced melting to help verify the model, it at least has an accepted theoretical basis.

We have simplified the GRAY model by assuming that there is a single melting line and that electronic contributions to $E_m(\eta)$ are negligible compared to lattice contributions. Therefore,
where $T_m$ is the melt temperature at $\eta = 1$.

Experimental data show that $G$ drops abruptly at melt from $\frac{1}{2} G_0$ to zero. However, because there are no similar data for $Y$, it is assumed to behave the same as $G$. Therefore, when $E$ exceeds $E_m(\eta)$, $Y$ and $G$ both become zero.

To minimize computer running time, Eqs. (4) and (A1) can be rewritten as simple polynomial functions of $\eta$.

APPENDIX B: BAUSCHINGER MODEL

An ideal work-hardening elastic-plastic material behaves elastically with shear modulus $G_0$ until it yields; then, at increasing stress levels, plastic flow occurs. When the direction of strain is reversed, the release path is again elastic until reverse yield occurs at the previous yield value. Although most metals load in an approximately ideal way, many show some departures, known as the Bauschinger effect, from the ideal unloading and reverse-loading curves. There are two important aspects of the Bauschinger effect. The first is that the release path is not elastic; the second, that the reverse flow stress does not reach the ideal yield value until considerable reverse flow has occurred. Thus, the actual release path may differ significantly from the ideal elastic-plastic release path. In the ideal case, plastic strain is totally irreversible, while elastic strain can be recovered. The Bauschinger effect removes this clear distinction. The Bauschinger effect can be attributed to dislocation interactions, formation of slip bands, twinning, or anisotropic grains. Whatever it may be, we wish only to find a reasonable way to represent the unloading paths.

A series of quasi-static measurements was conducted on 6061-T6 aluminum using a specially designed apparatus that subjected the test samples to uniaxial stress. The samples were initially loaded to strains of 1, 5, and 10% in both tension and compression; however, the behavior was identical for both cases. Data obtained for initial tensile loading were reduced to true deviator stress $S_1$ versus deviator strain $\varepsilon_1$.

To avoid the considerable complications in a description involving a Bauschinger yield model, the data were analyzed in terms of an effective shear modulus, allowing retention of the work-hardening yield model unmodified. The raw data are sufficiently precise to be differentiated. We can define an effective shear modulus as

$$G_{\text{eff}} = \frac{1}{2} \left( \frac{dS_1}{d\varepsilon_1} \right),$$

where $S_1$ is the deviator stress. Figure 2 is a plot of $G_{\text{eff}}$ normalized to $G_0$ versus $S_1/S_1^{\text{max}}$, where $S_1^{\text{max}}$ is the maximum value of $S_1$ upon loading. $S_1^{\text{max}}$ is related to the yield stress when unloading begins; it is two-thirds the yield strength in uniaxial strain. Within the scatter there appears to be a single curve for all three experiments. We do not maintain this interpretation is unique, but it does lead to a very simple model.

We generalize from the ideal elastic-plastic release path, with a constant shear modulus, to a curved path described by the variable effective shear modulus $G_{\text{eff}}(S_1/S_1^{\text{max}})$. If we maintain the yield model unmodified, this argument is bounded by $\pm 1$.

$G_{\text{eff}}$ is approximated by two linear splines (see Fig. 2). Therefore, to specify the path, only three values of $G_{\text{eff}}$ need to be given. These are

1. at first release, $G_{\text{eff}}(+1) = G_0$
2. at crossing the hydrostat (i.e., at $S_1 = 0$), $G_{\text{eff}}(0) = G_0$, and
3. at reverse yield, $G_{\text{eff}}(-1) = G_0$, with $G_0$ normally equal to 0.

Therefore, this is essentially a one-parameter model for the Bauschinger effect.

In addition, we found that for most materials, including aluminum, the model with $G_1 = G_0$ (i.e., no additional parameters) gave good agreement with experiment. This implies that for many materials the first aspect of the Bauschinger effect is small or absent. This conclusion is supported by other quasi-static measurements as well as plate impact experiments. In fact, this choice of parameters always appears to be a better approximation than the ideal elastic-plastic release path, yet involves no new parameters. A few materials, for example, uranium and titanium, show large effects during unloading with resulting small values for $G_1$.

A few other features must be added to specify the model completely. The path is independent of whether the initial load is tensile or compressive, in agreement with experiment. The effective shear modulus is never allowed to be less than the current slope of the work-hardening curve, as indicated qualitatively by the data. This ensures that reverse yield will be reached at reasonable strains, typically 4–10 times the elastic strain, at which point the Bauschinger release ends and the calculation returns to the normal constitutive model. Irreversibility is also maintained. If in the course of the Bauschinger path the strain rate is reversed, the shear modulus reverts to $G_0$. Finally, a small plastic-strain threshold must be exceeded before the model becomes operational.

More detailed information on this model may be found in Ref. 26.